On super edge-magic total labeling of banana trees

Muhammad Hussain¹, Edy Tri Baskoro², Slamin³

¹ School of Mathematical Sciences, GC University,
68-B New Muslim Town, Lahore, Pakistan
mhmaths@yahoo.com

² Combinatorial Mathematics Research Division,
Faculty of Mathematics and Natural Sciences,
Institut Teknologi Bandung
Jl. Ganesa 10 Bandung 40132, Indonesia
ebaskoro@math.itb.ac.id

³ Mathematics Education Study Program,
Universitas Jember
Jl. Kalimantan 37 Jember, Indonesia
slamin@unej.ac.id

Abstract. Let $G_1, G_2, ..., G_n$ be a family of disjoint stars. The tree obtained by joining a new vertex $a$ to one pendant vertex of each star $G_i$ is called a banana tree. In this paper we determine the super edge-magic total labelings of the banana trees that have not been covered by the previous results [11].

Keywords: Super edge-magic total labeling, banana tree.

1 Introduction

All graphs in this paper are finite, simple and undirected. The graph $G$ has the vertex-set $V(G)$ and edge-set $E(G)$. A general reference for graph-theoretic ideas can be seen in [12].

A labeling (or valuation) of a graph is a map that carries graph elements to numbers (usually to positive or non-negative integers). In this paper the domain will usually be the set of all vertices and edges and such labelings are called total labelings. Some labelings use the vertex-set only, or the edge-set only, and we shall call them vertex-labelings and edge-labelings respectively. Other domains are possible. The most complete recent survey of graph labelings can be seen in [5]. There are many types of graph labelings, for example harmonius, cordial, graceful and antimagic. In this paper, we focus on one type of labeling called edge-magic total labeling.
Definition 1. An edge-magic total labeling of a graph $G$ is a one-to-one map $\lambda$ from $V(G) \cup E(G)$ onto the integers $\{1, 2, \cdots, |V(G) \cup E(G)|\}$ with the property that, there is an integer constant $k$ such that $\lambda(x) + \lambda(x, y) + \lambda(y) = k$ for any $(x, y) \in E(G)$.

It will be convenient to call $\lambda(x) + \lambda(x, y) + \lambda(y)$ the edge sum of $(x, y)$, and $k$ the constant magic of $G$.

Definition 2. An edge-magic total labeling $\lambda$ of graph $G$ is called super edge-magic total labeling if $\lambda(V(G)) = \{1, 2, \cdots, |V(G)|\}$.

A number of classification studies on edge magic total graphs has been intensively investigated. A part of these studies’ results include

- Every cycle $C_n$ is super edge-magic if and only if $n$ is odd [2].
- $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$ [2].
- $K_n$ is super edge-magic if and only if $n = 1, 2, or 3$ [2].
- $nK_2$ is edge-magic when $n$ is odd, actually shows that it is super edge-magic [6].
- $2P_n$ is super edge-magic if and only if $n$ is not 2 or 3 , $2P_{4n}$ is super edge-magic for all $n$ [13].
- The friendship graph consisting of $n$ triangles is super edge-magic if and only if $n$ is 3, 4, 5 or 7 [10].
- $nP_3$ is super edge-magic for $n \geq 4$ and $n$ even [1].
- The fan $F_n$ is super edge-magic if and only if $1 \leq n \leq 6$ [3].

However, Enomoto et al. [2] conjectured that every tree admits a super edge-magic total labeling. In the effort of attacking this conjecture, many authors have considered super edge-magic total labeling for some particular classes of trees. Lee and Shah [8] have verified this conjecture for trees on at most 17 vertices with a computer help. Earlier, in [6] Kotzig and Rosa proved that every caterpillar is super edge-magic. However, this conjecture still remains open. The super edge-magic total labeling of other class of trees, such as a banana tree has partially been discovered. The definition of banana tree is given below.

Definition 3. Let $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_k}$ be a family of disjoint stars with the vertex-sets $V(K_{1,n_i}) = \{c_i, a_{i1}, \ldots, a_{in_i}\}$ and $\deg(c_i) = n_i$, $1 \leq i \leq k$. A banana tree $BT(n_1, n_2, \ldots, n_k)$ is a tree obtained by adding a new vertex $a$ and joining it with $a_{11}, a_{21}, \ldots, a_{k1}$.
Swaminathan and Jeyanthi [11] proved some results on super edge-magic total labeling of banana trees. These results are as follows:

- $BT(n_1, n_2, ..., n_k)$, $n_j \geq j$, $2 \leq j \leq k$ admits a super edge-magic labeling, where $n_1$ is any positive integer.
- In particular, if $n_1 = n_2 = ... = n_k = n \geq k$, then the smallest magic constant of $BT(n_1, n_2, ..., n_k)$ is $2nk + 3k + 4$.
- $BT(n, 1, 1)$ admits a super edge-magic labeling with the smallest magic constant $2n+16$.

In this paper we present super edge-magic total labelings of banana trees with some conditions that have not been considered in [11], namely we prove that:

- $BT(n_1, n_2, ..., n_k)$ admits a super edge-magic labeling if $n_1 = n_2 = ... = n_k = n$, $\lceil \frac{k}{2} \rceil \leq n \leq k - 1$, and
- $BT(n_1, n_2, ..., n_k)$ admits a super edge-magic labeling if $n_1 > n_2 > ... > n_k > 1$.

We also consider a disjoint union of banana trees and prove that:

- For $m \geq 2$, $n \geq 2m$, $G \cong mBT(n, n)$ admits a super edge-magic labeling, and
- $H \cong 2BT(n_1, n_2, ..., n_k)$ admits a super edge-magic labeling if $n_1 = n_2 = ... = n_k = n; n \geq 2k, k \geq 3$. 

Fig. 1. A banana tree $BT(n_1, n_2, ..., n_k)$. 
2 Main Results

Before giving our main results, let us consider the following lemma found in [3] that gives a necessary and sufficient condition for a graph to be super edge-magic.

**Lemma 1.** A graph $G$ with $p$ vertices and $q$ edges is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the set $S = \{f(x) + f(y) | xy \in E(G)\}$ consists of $q$ consecutive integers. In such a case, $f$ extends to a super edge-magic total labeling of $G$ with magic constant $c = p + q + s$, where $s = \min(S)$ and

$$S = \{f(x) + f(y) | xy \in E(G)\} = \{c - (p + 1), c - (p + 2), \ldots, c - (p + q)\}.$$

For Theorems 1 and 2, let denote the vertex and edge sets of $BT(n_1, n_2, ..., n_k)$ as follows:

$V = \{a\} \cup \{c_i | 1 \leq i \leq k\} \cup \{a_{ij} | 1 \leq j \leq n; 1 \leq i \leq k\},$

$E = \{aa_{i1} | 1 \leq i \leq k\} \cup \{c_ia_{ij} | 1 \leq j \leq n; 1 \leq i \leq k\}.$

**Theorem 1.** $G \cong BT(n_1, n_2, ..., n_k)$ admits a super edge-magic labeling if $n_1 = n_2 = ... = n_k = n$, $\lceil \frac{k}{2} \rceil \leq n \leq k - 1$.

**Proof.** If $v=|V(G)|$ and $e=|E(G)|$ then

$$v = n_1 + n_2 + ... + n_k + k + 1 = nk + k + 1,$$

$$e = n_1 + n_2 + ... + n_k + k = nk + k.$$

Now, construct a labeling $\lambda : V \cup E \rightarrow \{1, 2, ..., v + e\}$ as follows:

$$\lambda(a) = (n + 1)k + 1 - \left\lfloor \frac{k}{2} \right\rfloor,$$

$$\lambda(c_i) = \begin{cases} 
  nk + i, & \text{for } 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \\
  nk + 1 + i, & \text{for } \left\lceil \frac{k}{2} \right\rceil < i \leq k.
\end{cases}$$

$$\lambda(a_{i1}) = \begin{cases} 
  (n + 1)i - \left\lfloor \frac{k}{2} \right\rfloor, & \text{for } i \leq \left\lfloor \frac{k}{2} \right\rfloor \\
  (n + 1)i - n - \left\lfloor \frac{k}{2} \right\rfloor, & \text{for } i \geq \left\lfloor \frac{k}{2} \right\rfloor + 1.
\end{cases}$$
The set of all edge-sums generated by the above formula forms a consecutive integer sequence $nk + 2, nk + 3, \ldots, nk + (n + 1)k + 1$. Therefore by Lemma 1 $\lambda$ can be extended to a super edge-magic labeling and we obtain the magic constant $h = v + e + s = (nk + k + 1) + (nk + k) + nk + 2 = 3nk + 2k + 3$. $\square$

**Theorem 2.** $G \cong BT(n_1, n_2, \ldots, n_k)$ admits a super edge-magic labeling if $n_1 > n_2 > \ldots > n_k > 1$.

**Proof.** If $v = |V(G)|$ and $e = |E(G)|$ then clearly

$$v = n_1 + n_2 + \ldots + n_k + k + 1,$$

$$e = n_1 + n_2 + \ldots + n_k + k.$$

Now we will define labeling $\lambda$

$$\lambda : V \cup E \to \{1, 2, \ldots, v + e\}$$

as follows

$$\lambda(a) = v - 2$$

$$\lambda(c_i) = \begin{cases} (p - k - 1) + i, & \text{for } 1 \leq i \leq k - 2 \\ (p - k) + i, & \text{for } k - 1 \leq i \leq k. \end{cases}$$

$$\lambda(a_{i1}) = \begin{cases} \sum_{t=1}^{i} n_t - (k - i - 2), & \text{for } 1 \leq i \leq k - 2 \\ \sum_{t=1}^{i+1} n_t - (n_i + 1), & \text{for } i = k - 1 \\ \sum_{t=1}^{i} n_t, & \text{for } i = k \end{cases}$$
\[
\lambda(\{a_{ij} \mid 2 \leq j \leq n_i\}) = \begin{cases} 
\{(\sum_{i=1}^{i} n_t + (1 - t) \mid 1 \leq t \leq n_i) \backslash \{\sum_{i=1}^{i} n_t - (k - i - 2)\} \} & \text{for } 1 \leq i \leq k - 2 \\
\{(\sum_{i=1}^{i} n_t + (1 - t) \mid 1 \leq t \leq n_i) \backslash \{\sum_{i=1}^{i+1} n_t - (n_i + 1)\} \} & \text{for } i = k - 1 \\
\{(\sum_{i=1}^{i} n_t + (1 - t) \mid 1 \leq t \leq n_i) \backslash \{\sum_{i=1}^{i} n_t\} \} & \text{for } i = k 
\end{cases}
\]

The set of all edge-sums generated by the above formula forms a consecutive integer sequence \(v + 1 - k, v + 2 - k, ..., v + e - k\). Therefore by Lemma 1 \(\lambda\) can be extended to a super edge-magic labeling and we obtain the magic constant \(h = v + e + s = (n_1 + n_2 + ... + n_k + k + 1) + (n_1 + n_2 + ... + n_k + k) + v + 1 - k = v + v - 1 + v + 1 - k = 3v - k\).

\textbf{Theorem 3.} For \(m \geq 2, n \geq 2m\), \(G \cong mBT(n, n)\) admits a super edge-magic labeling.

\textbf{Proof.} Let denote the vertices and edges of \(G\), as follows:

\[V(G) = \{a_i \mid 1 \leq i \leq m\} \cup \{a_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq 2\} \cup \{a_{ij}^k \mid 1 \leq i \leq m; 1 \leq j \leq 2; 1 \leq k \leq n\},\]

\[E(G) = \{a_i a_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq 2\} \cup \{a_{ij} a_{ij}^k \mid 1 \leq i \leq m; 1 \leq j \leq 2, 1 \leq k \leq n\}.\]

If \(v = |V(G)|\) and \(e = |E(G)|\) then

\[v = n(n + 1) + m,\]

\[e = n(n + 1).\]

Now construct a labeling \(\lambda: V \cup E \rightarrow \{1, 2, ..., v + e\}\) as follows:

\[\lambda(a_i) = v - m + i, \quad 1 \leq i \leq m\]

\[\lambda(a_{ij}) = \lambda(a_i) + i + j - (2m + 2), \quad 1 \leq i \leq m, 1 \leq j \leq 2\]

\[\lambda(a_{i1}) = (n - 2m + 1) + (2n + 1)(i - 1), \quad 1 \leq i \leq m\]
\(\lambda(a_{i1}^1) = (2n - 2m + 2) + (2n + 1)(i - 1), \quad 1 \leq i \leq m\)

\(\lambda(\{a_{i1}^k \mid 2 \leq k \leq n\}) = \{(2i-2)n+t \mid 1 \leq t \leq n\} \setminus \{\lambda(a_{i1})\}, \quad \text{for} \quad 1 \leq i \leq m\)

\(\lambda(\{a_{i2}^k \mid 2 \leq k \leq n\}) = \{(2i-1)n+t \mid 1 \leq t \leq n\} \setminus \{\lambda(a_{i2})\}, \quad \text{for} \quad 1 \leq i \leq m\)

The set of all edge-sums generated by the above formula forms a consecutive integer sequence \(2nm + 2, 2nm + 3, \cdots, 2nm + 2(n + 1) + 1\). Therefore by Lemma 1 \(\lambda\) can be extended to a super edge-magic labeling and we obtain the magic constant \(h = v + e + s = n(n + 1) + m + n(n + 1) + 2nm + 2 = 2n(n + 1) + 2nm + m + 2\). \(\square\)

**Example 1.** Super edge-magic total labeling of 3BT(6, 6) is given in Figure 2.

![Fig. 2.](image)

**Theorem 4.** \(H \cong 2BT(n_1, n_2, ..., n_k)\) admits a super edge-magic labeling if \(n_1 = n_2 = ... = n_k = n; \quad n \geq 2k, \quad k \geq 3\).

**Proof.** Let denote the vertices and edges of \(H\), as follows:

\(V(H) = \{a_i \mid 1 \leq i \leq 2\} \cup \{a_{ij} \mid 1 \leq i \leq 2, 1 \leq j \leq k\} \cup \{a_{ij}^l \mid 1 \leq i \leq 2, 1 \leq j \leq k, 1 \leq l \leq n\}\),

\(E(H) = \{a_i a_{ij}^1 \mid 1 \leq i \leq 2, 1 \leq j \leq k\} \cup \{a_{ij} a_{ij}^l \mid 1 \leq i \leq 2, 1 \leq j \leq k, 1 \leq l \leq n\}\).

If \(v = |V(G)|\) and \(e = |E(G)|\) then

\(v = 2k(n + 1) + m,\)

\(e = 2k(n + 1).\)

Now, construct a labeling \(\lambda : V \cup E \rightarrow \{1, 2, ..., v + e\}\) as follows:
\[ \lambda(a_i) = v - m + i, \quad 1 \leq i \leq 2 \]
\[ \lambda(a_{1j}) = \lambda(a_1) - 2k + (j - 1), \quad 1 \leq j \leq k \]
\[ \lambda(a_{2j}) = \lambda(a_2) - k - 2 + j, \quad 1 \leq j \leq k \]
\[ \lambda(a_{1j}) = \sum_{j=1}^{k} n_j - 2k + j \]
\[ \lambda(a_{2j}) = \sum_{j=1}^{k} n_j + (2m + 1)k + (j - 1) \]
\[ \lambda(\{a_{lj} \mid 2 \leq l \leq n\}) = ((j-1)n+t \mid 1 \leq t \leq n \} \{\lambda(a_{lj})\} , \text{for } 1 \leq j \leq k \]
\[ \lambda(\{a_{lj} \mid 2 \leq l \leq n\}) = \{kn+(j-1)n+t \mid 1 \leq t \leq n \} \{\lambda(a_{lj})\} , \text{for } 1 \leq j \leq k \]

The set of all edge-sums generated by the above formula forms a consecutive integer sequence \(2nk+2, 2nk+3, \ldots, 2nk+2k(n+1)+1\).
Therefore by Lemma 1 \(\lambda\) can be extended to a super edge-magic labeling and we obtain the magic constant \(h = v + e + s = 2k(n + 1) + m + 2k(n + 1) + 2nk + 2 = 6nk + 4k + m + 2\). \qed

**Example 2.** Super edge-magic total labeling of \(2BT(8,8,8,8)\) is given in Figure 3.

![Figure 3](image-url)
References

8. S. M. Lee, and Q. X. Shah, All trees with at most 17 vertices are super edge-magic, 16th MCCCC Conference, Carbondale, University Southern Illinois, Nov. 2002.